On Average Baby PIH and Its Applications

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Constraint Satisfaction Problem (qCSP)

- Variables $X = \{x_1, \cdots, x_n\}$
- Alphabet Σ
- Constraints $\Phi = \{\varphi_1, \cdots, \varphi_m\}$, each depends on q variables

- Decide: whether it's satisfiable or not.

NP-Complete.

The PCP Theorem [AS-ALMSS'98] [Dinur'07]

- **NP-hard** to decide whether a qCSP instance is
  - Satisfiable, or
  - Cannot satisfy s-fraction of constraints simultaneously.

$(0 < s < 1)$

Relaxation: Multi-Assignment

- Assign each variable a **set** of values.

x_1: { 1, 5, 7, 9 }

x_2: { 2, 3, 4 }

x_3: { 2, 6 }

x_4: { 4, 5, 6, 8 }

$\varphi_1 = (x_1 x_2, C_1)$

$\varphi_2 = (x_2 x_3, C_2)$

$\varphi_3 = (x_2 x_4, C_3)$

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Multi-Assignment PCP [Arora,Moshkovitz,Safra'06]

- **NP-hard** to decide whether a qCSP instance is
  - Satisfiable, or
  - Cannot satisfy s-fraction of constraints simultaneously **even** when **each** variable assigned $\leq t$ values.

$$(0 < s < 1,\ t > 1)$$

- Used to prove NP-hardness of approximating SetCover.

Parameterized Inapprox. Hypo. (PIH)

- Hypothesis[Lokshtanov,Ramanujan,Saurabh,Zehavi'20]:

  No FPT algorithm decide a 2CSP parameterized by $k = |X|$ is:
  - Satisfiable, or
  - Cannot satisfy s-fraction of constraints simultaneously. $(0 < s < 1)$

- SOTA: Exponential Time Hypothesis -> PIH. [Guruswami,Lin,Ren,Sun,Wu'24]

- Major open problem: W[1]≠FPT -> PIH ?

Weaken: Baby PIH [Guruswami,Ren,Sandeep'24]

- No FPT algorithm for deciding a 2CSP parameterized by $k = |X|$:
  - Being satisfiable, or
  - Cannot satisfy all constraints simultaneously even when **each** variable assigned $\leq t$ values. $(t > 1)$

- W[1]≠FPT -> Baby PIH. [Guruswami,Ren,Sandeep'24]
  - Following the method in [Barto,Kozik'22] showing Baby PCP without using PCP Theorem.

Weaken: Baby PIH [Guruswami,Ren,Sandeep'24]

x_1: $\{0,\ 1,\ 3,\ 5,\ 7,\ 8,\ 9\}$

x_2: $\{2\}$

x_3: $\{2,\ 6\}$

x_4: $\{8\}$

$\longleftarrow \leq t \longrightarrow$

$(t = 7)$

Question: Average Baby PIH

$|X| = 4,$

x_1: $\{0,\ 1,\ 3,\ 5,\ 7,\ 8,\ 9\}$

x_2: $\{2\}$

x_3: $\{2,\ 6\}$

x_4: $\{8\}$

$\longleftarrow \le t \longrightarrow$

$(t = 7)$

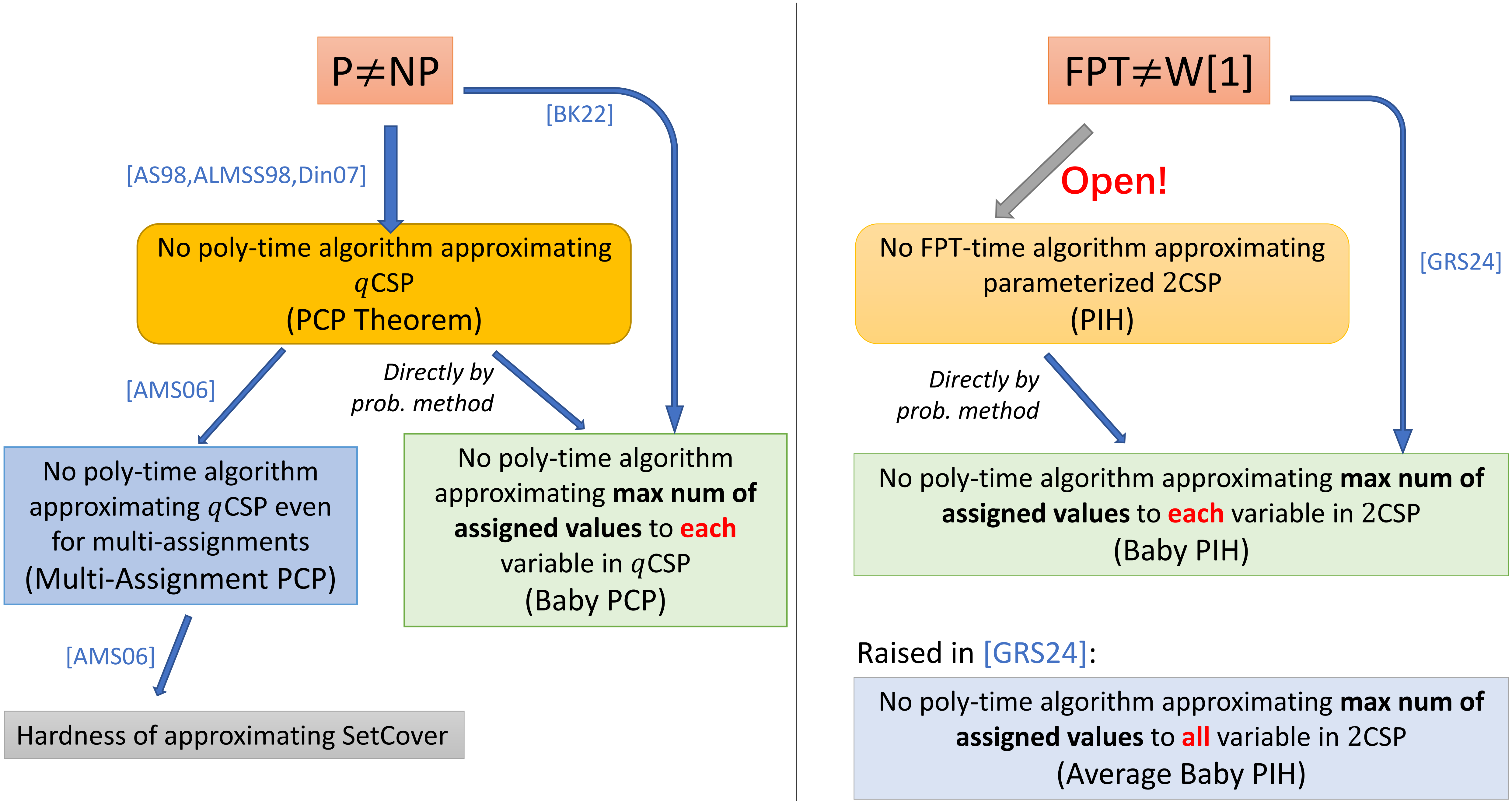
Total # of values: $7 + 1 + 2 + 1 = 11 = 2.75|X|$.

Question: Average Baby PIH

- No FPT algorithm for deciding a 2CSP parameterized by $k = |X|$:
  - Being satisfiable, or
  - Cannot satisfy all constraints simultaneously even when assigning to X less than $t|X|$ values **in total**. $(t > 1)$

ℓ_1 instead of ℓ_∞

- Raised in [Guruswami,Ren,Sandeep'24].



P≠NP
[AS98,ALMSS98,Din07]
[BK22]
No poly-time algorithm approximating qCSP (PCP Theorem)
[AMS06]
Directly by prob. method
No poly-time algorithm approximating qCSP even for multi-assignments (Multi-Assignment PCP)
No poly-time algorithm approximating max num of assigned values to each variable in qCSP (Baby PCP)
[AMS06]
Hardness of approximating SetCover
FPT≠W[1]
Open!
[GRS24]
No FPT-time algorithm approximating parameterized 2CSP (PIH)
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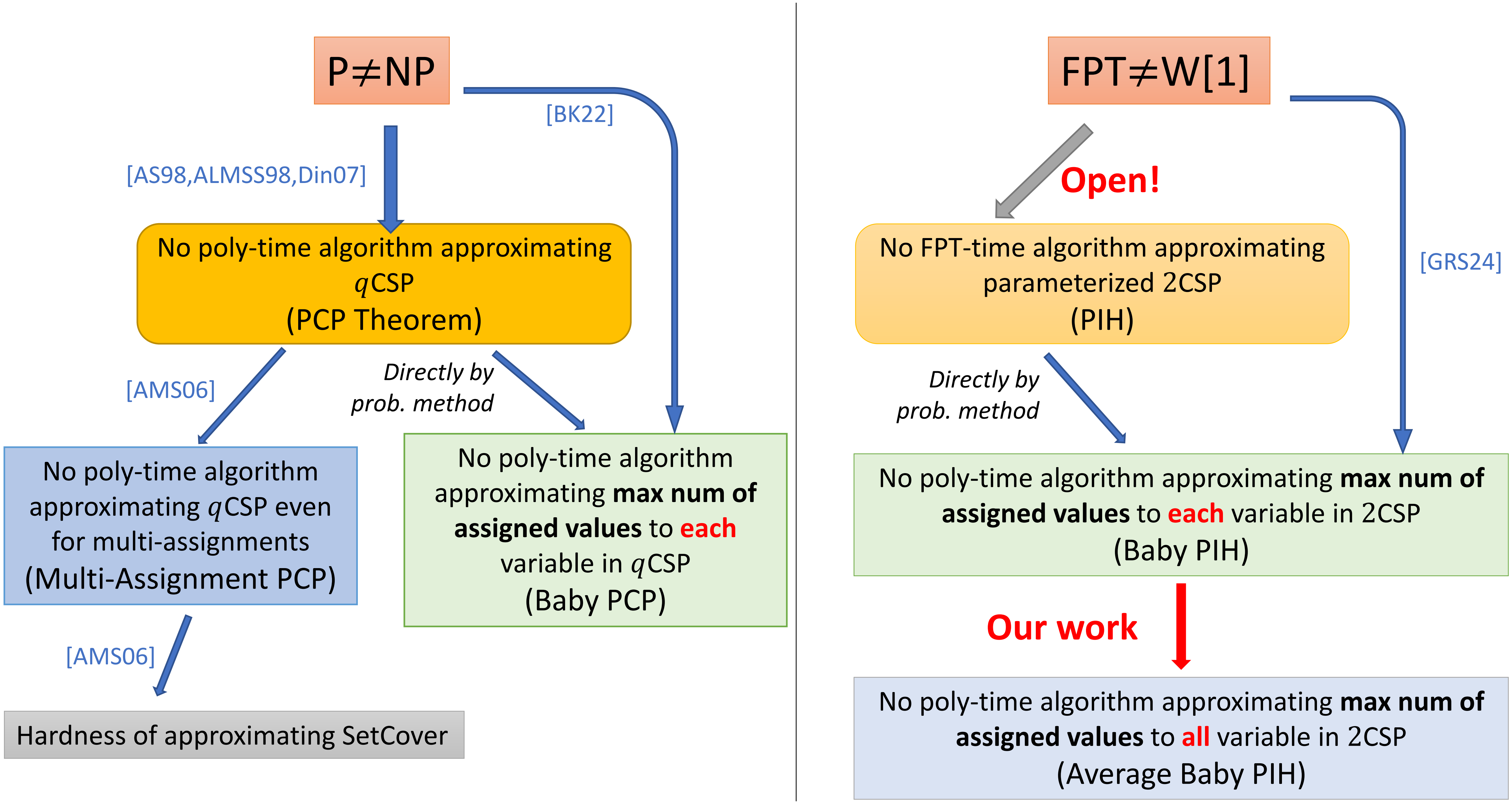
No poly-time algorithm approximating max num of assigned values to each variable in 2CSP (Baby PIH)
Raised in [GRS24]:
No poly-time algorithm approximating max num of assigned values to all variable in 2CSP (Average Baby PIH)

Our result

W[1]$\neq$FPT ⟶ Average Baby PIH



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Our work
No poly-time algorithm approximating max num of
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W[1]≠FPT → Average Baby PIH

- A reduction for 2CSP instances that:

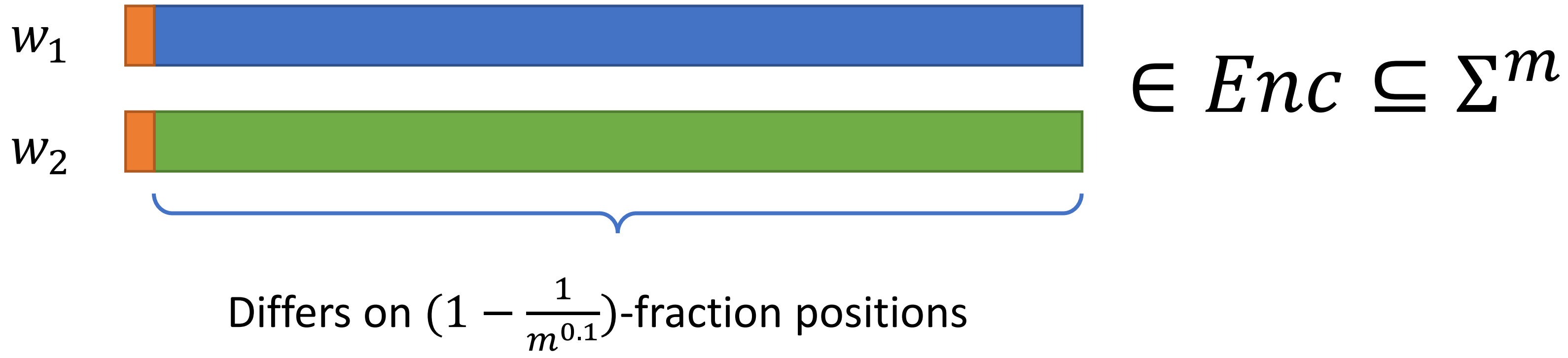
Satisfiable	→	Satisfiable
Can't satisfied when **each** variable assigned $\leq t$ values.	→	Can't satisfied when assigning to X less than $\frac{t}{2}\lvert X\rvert$ values **in total**.

Baby PIH —Our work→ Average Baby PIH

W[1]≠FPT → Baby PIH [Guruswami,Ren,Sandeep'24]

Technical Tool

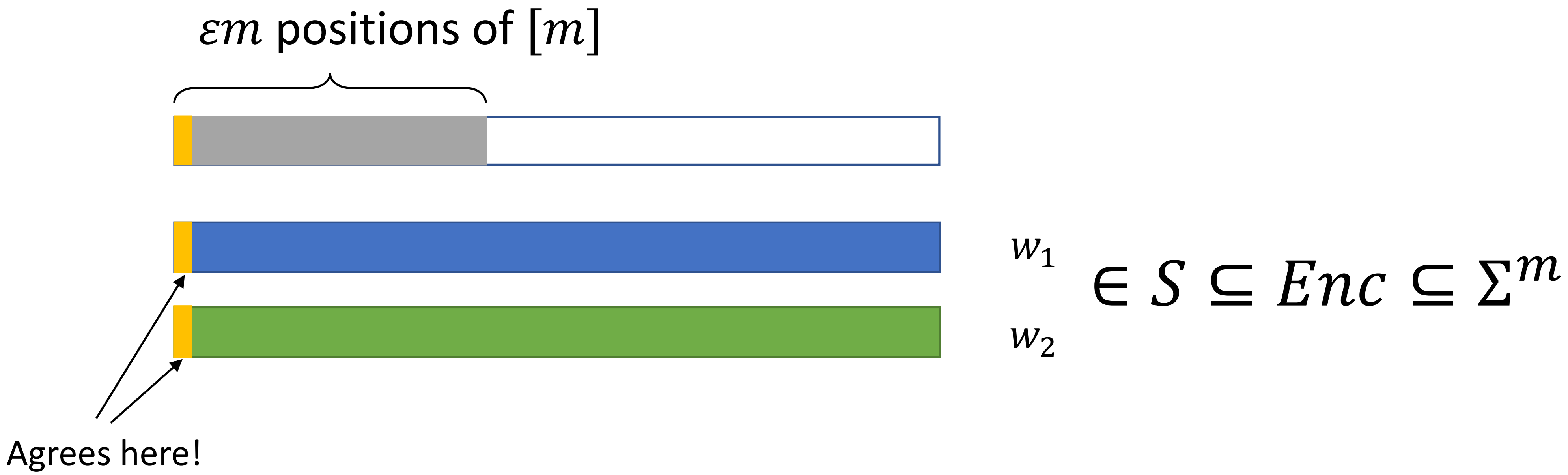
- Error-correcting codes with ***overwhelming*** (relative) distance



e.g. Reed-Solomon codes.

Technical Tool

Any set S of codewords that "collides" on a noticeable fraction of positions......



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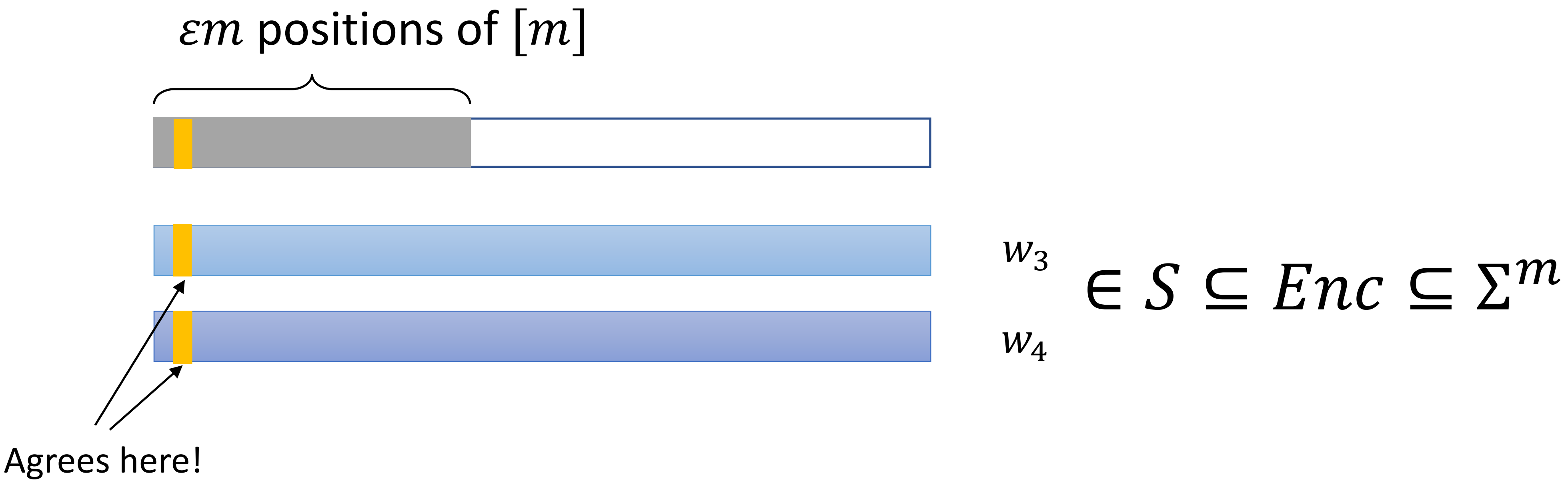

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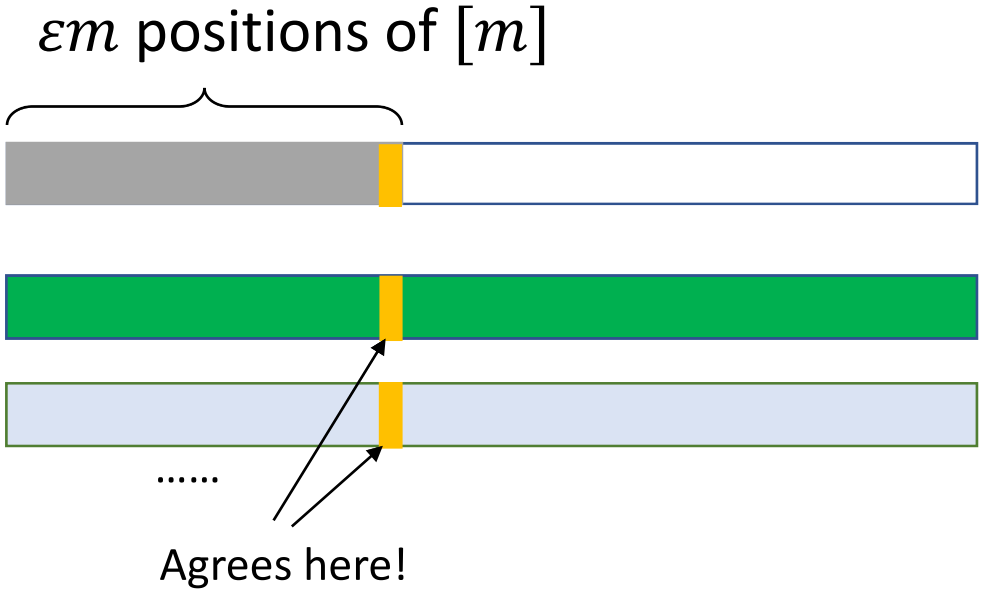
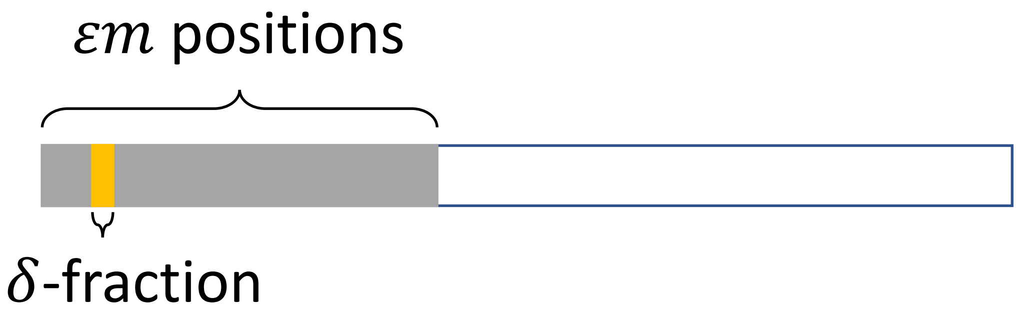


w_s, w_{s+1} $\in S \subseteq Enc \subseteq \Sigma^m$

Technical Tool



Theorem(Informal) cf. [Karthik-Navon'21, Lin-Ren-Sun-Wang'23]:
For code Enc with relative distance $1 - \delta$, any set of codewords "collides" on εm positions must have size $\geq \sqrt{\frac{2\varepsilon}{\delta}}$.

Recall: W[1]≠FPT → Average Baby PIH

- A reduction for 2CSP instances that:

Satisfiable	→	Satisfiable
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Baby PIH —Our work→ Average Baby PIH

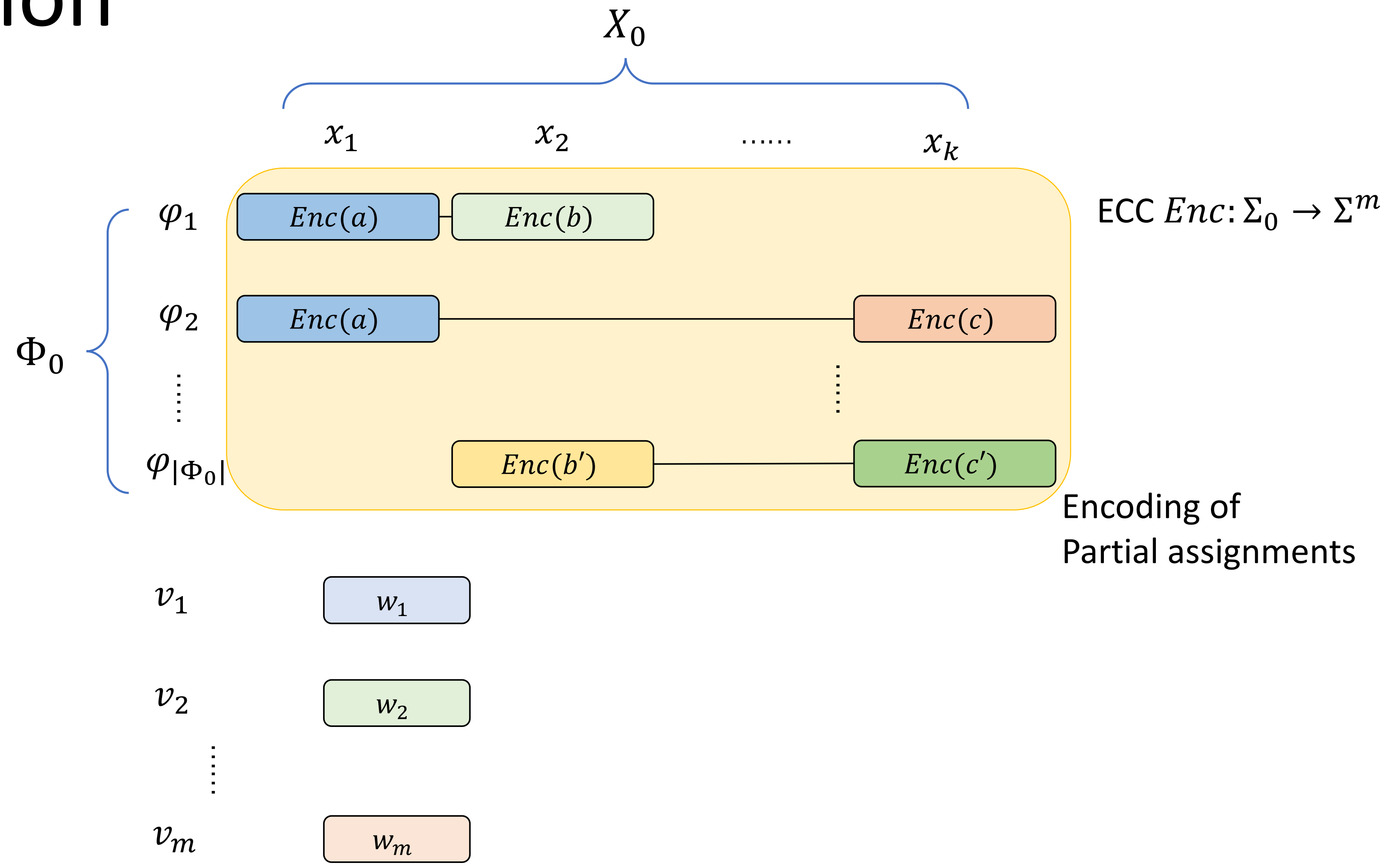
W[1]≠FPT → Baby PIH [Guruswami,Ren,Sandeep'24]

The Reduction

Input: 2CSP instance
$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$

Output: 2CSP instance Π
as shown.

Variables: $\Phi_0 \cup \{v_1, \ldots, v_m\}$



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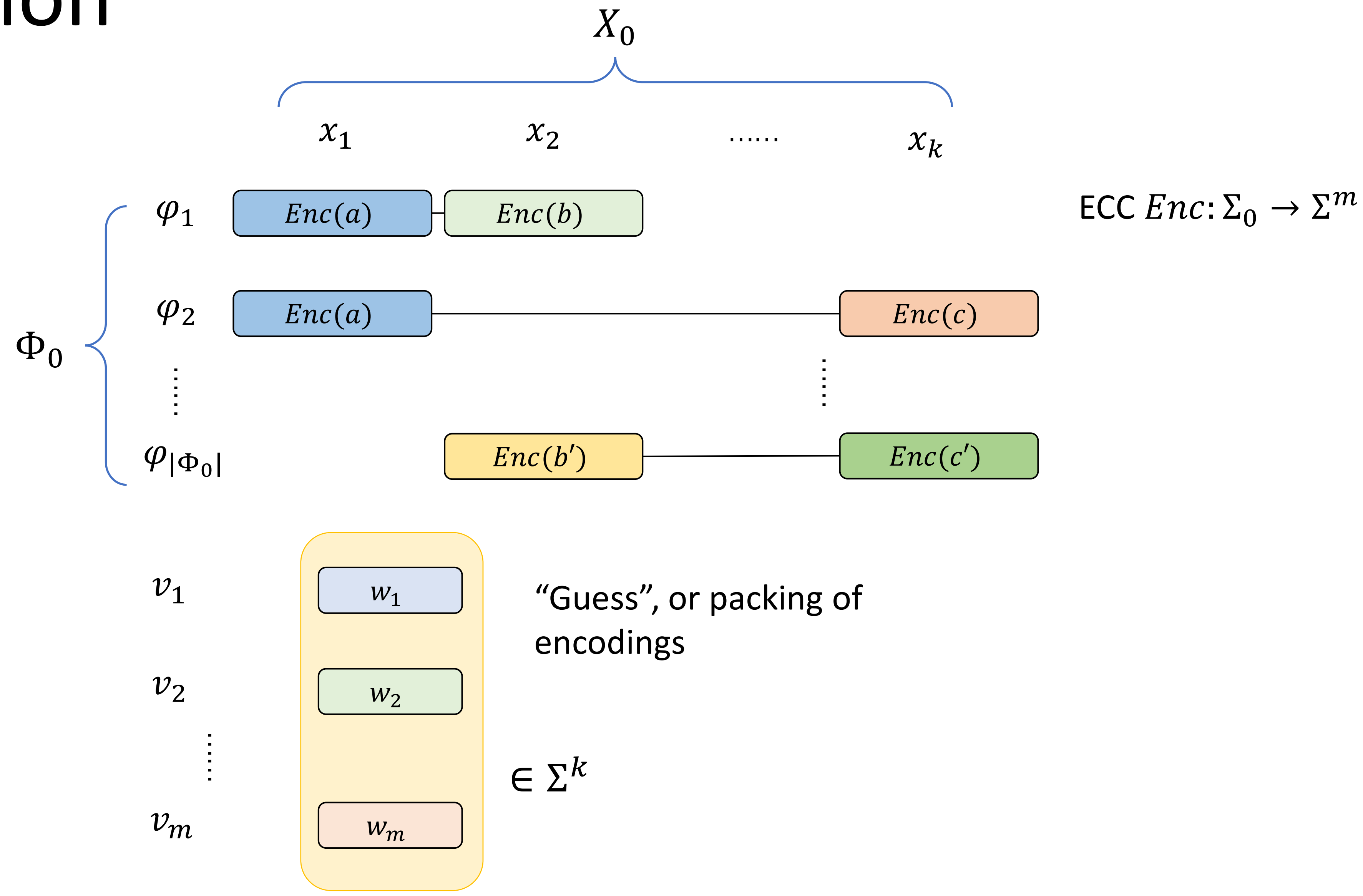

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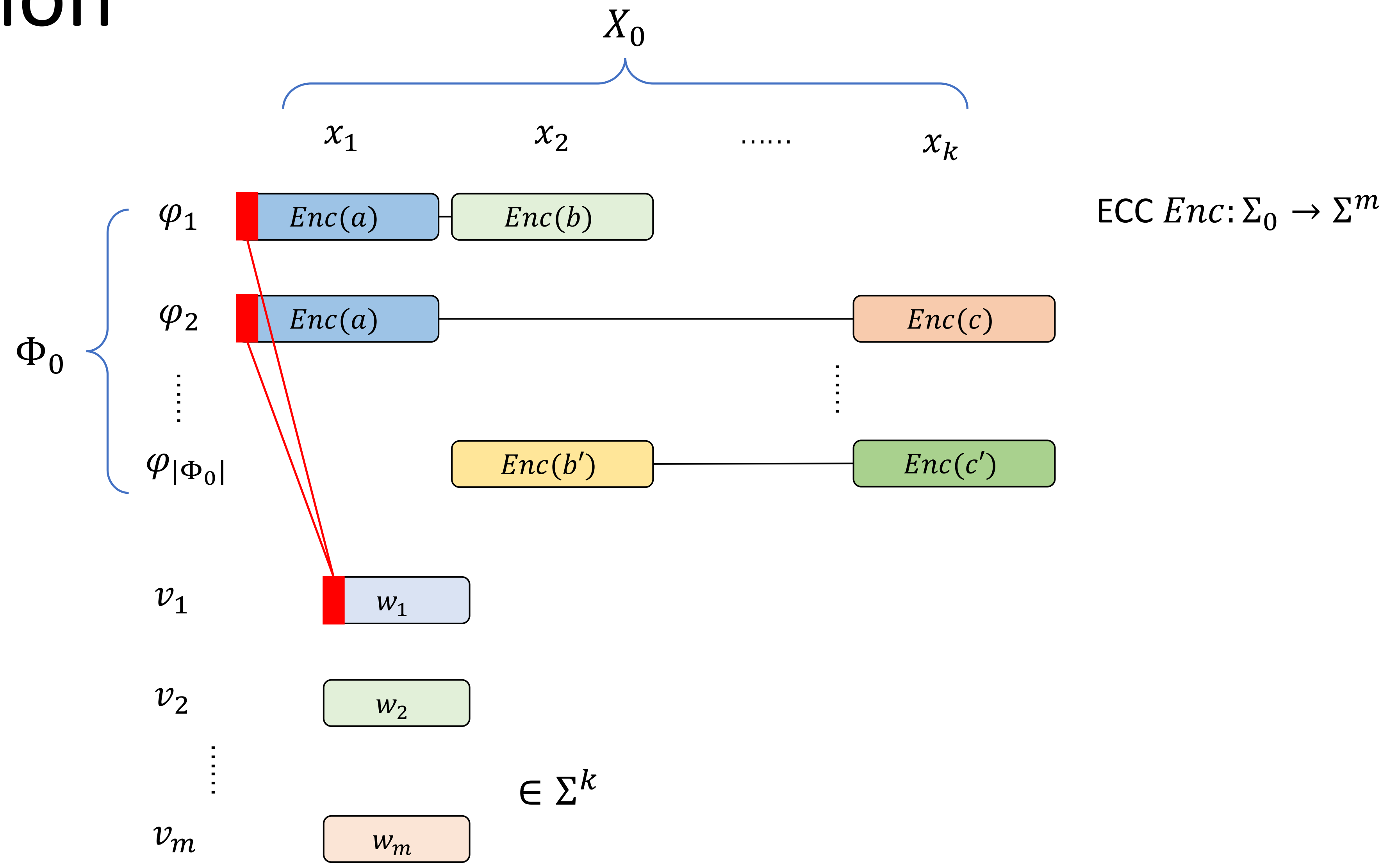
Constraints: Equality Check



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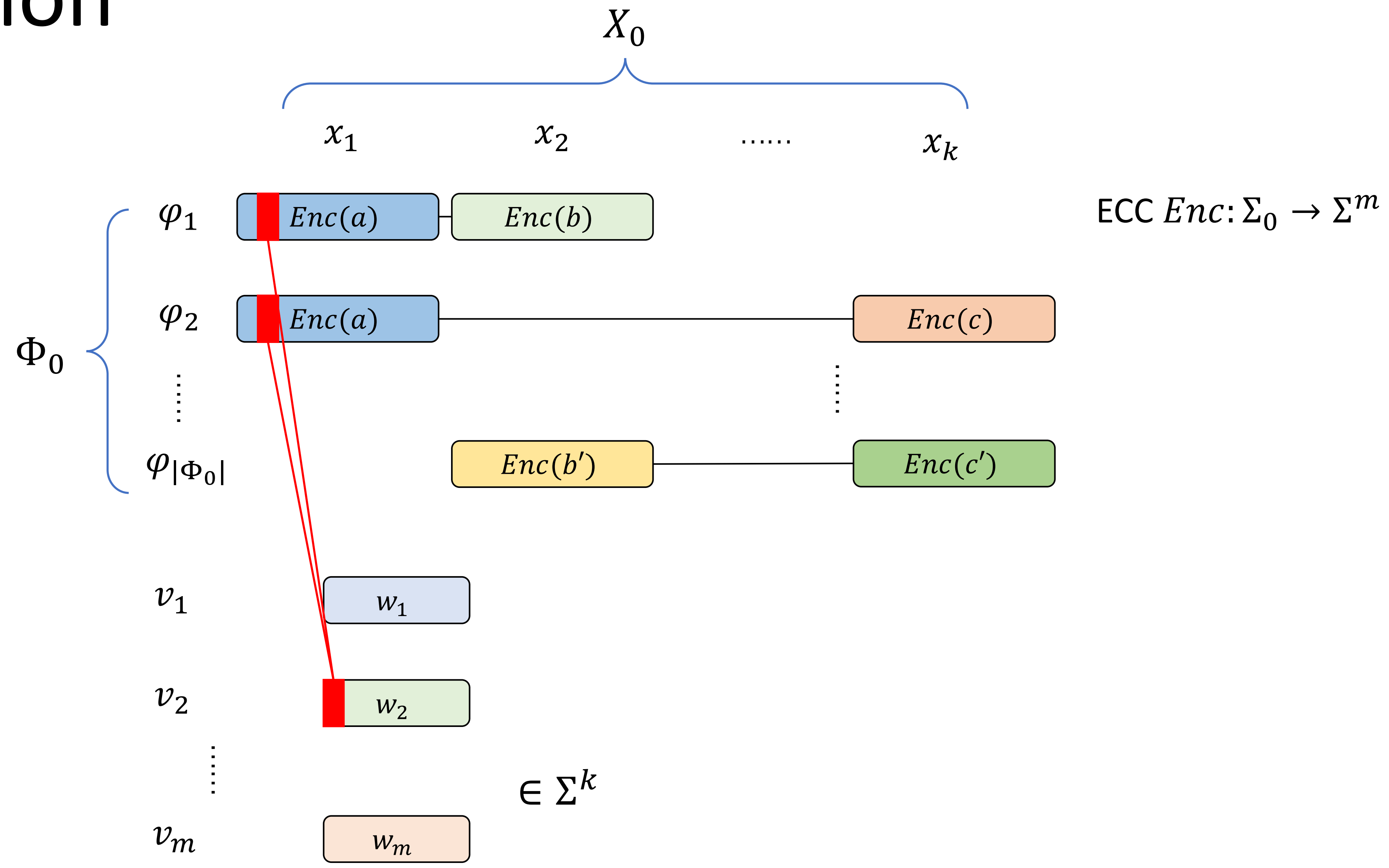

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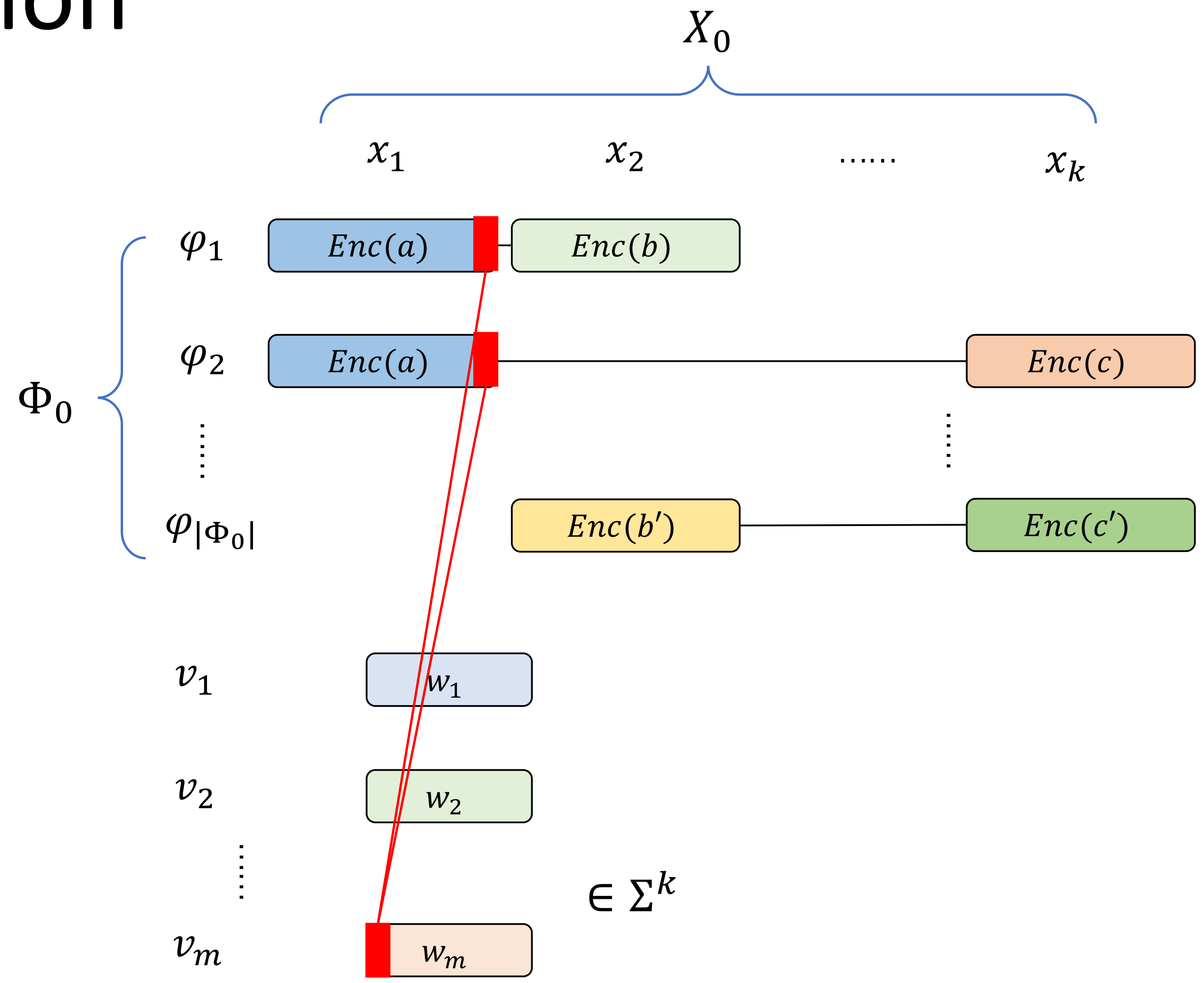

ECC $Enc: \Sigma_0 \to \Sigma^m$

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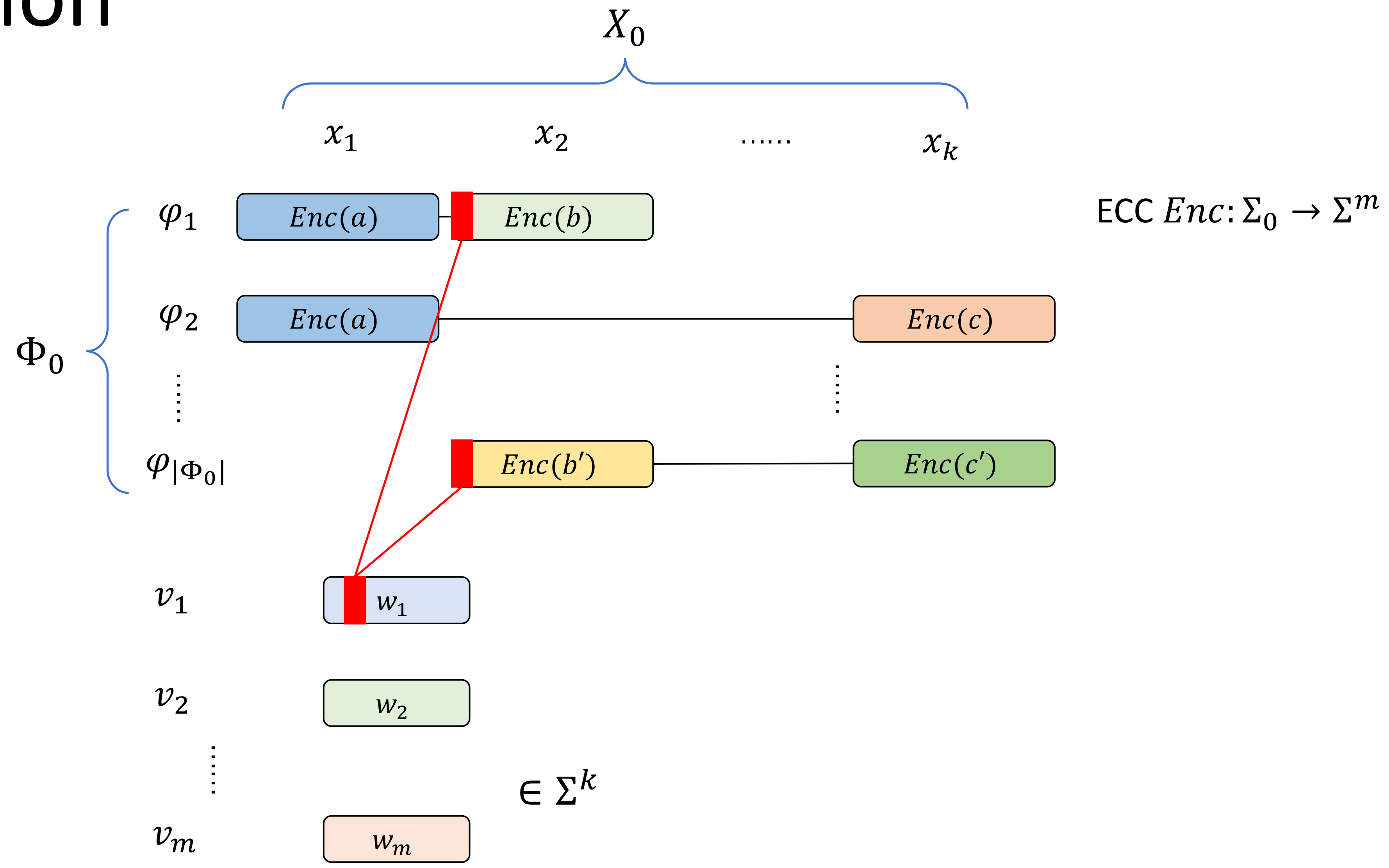

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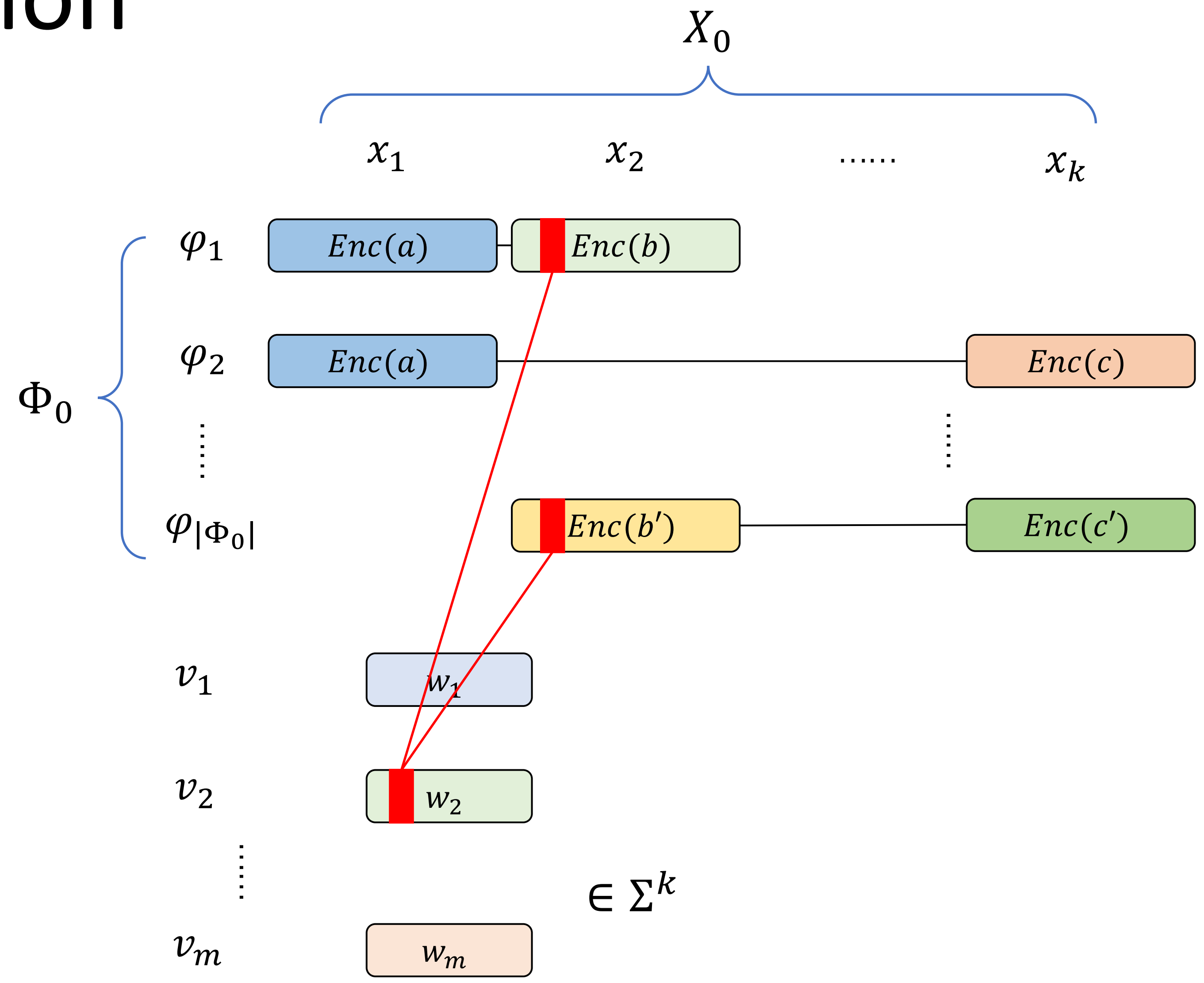

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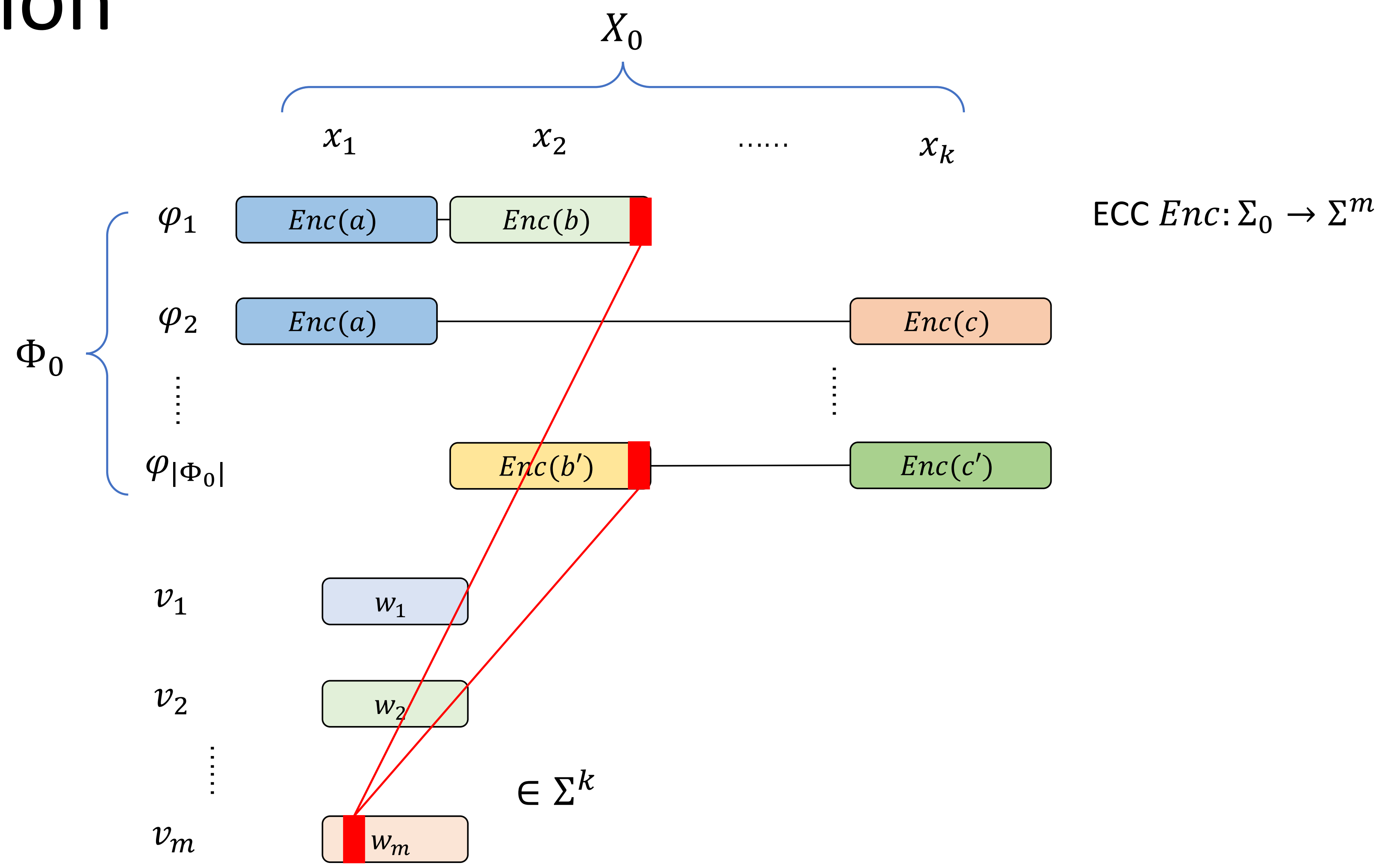

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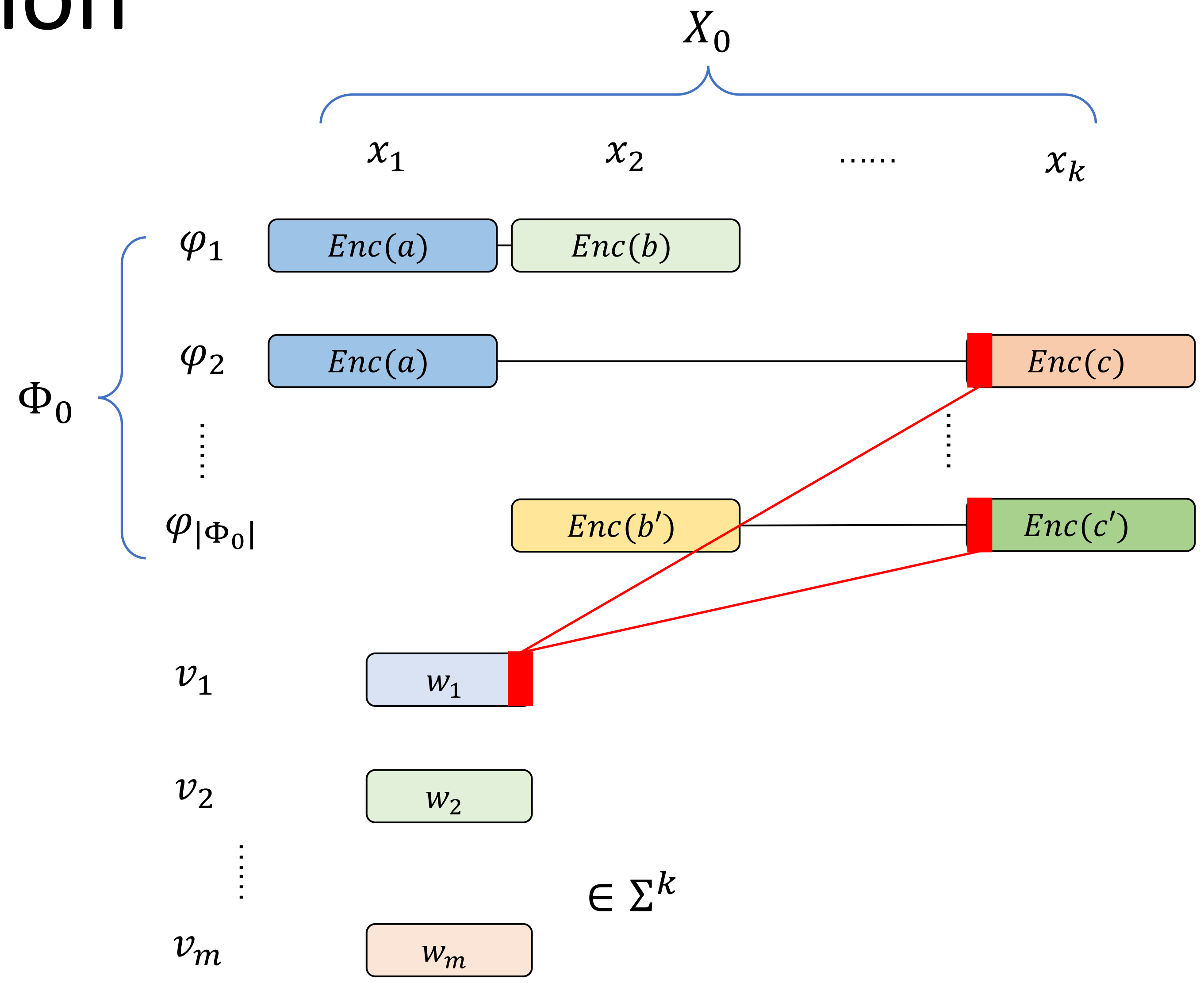

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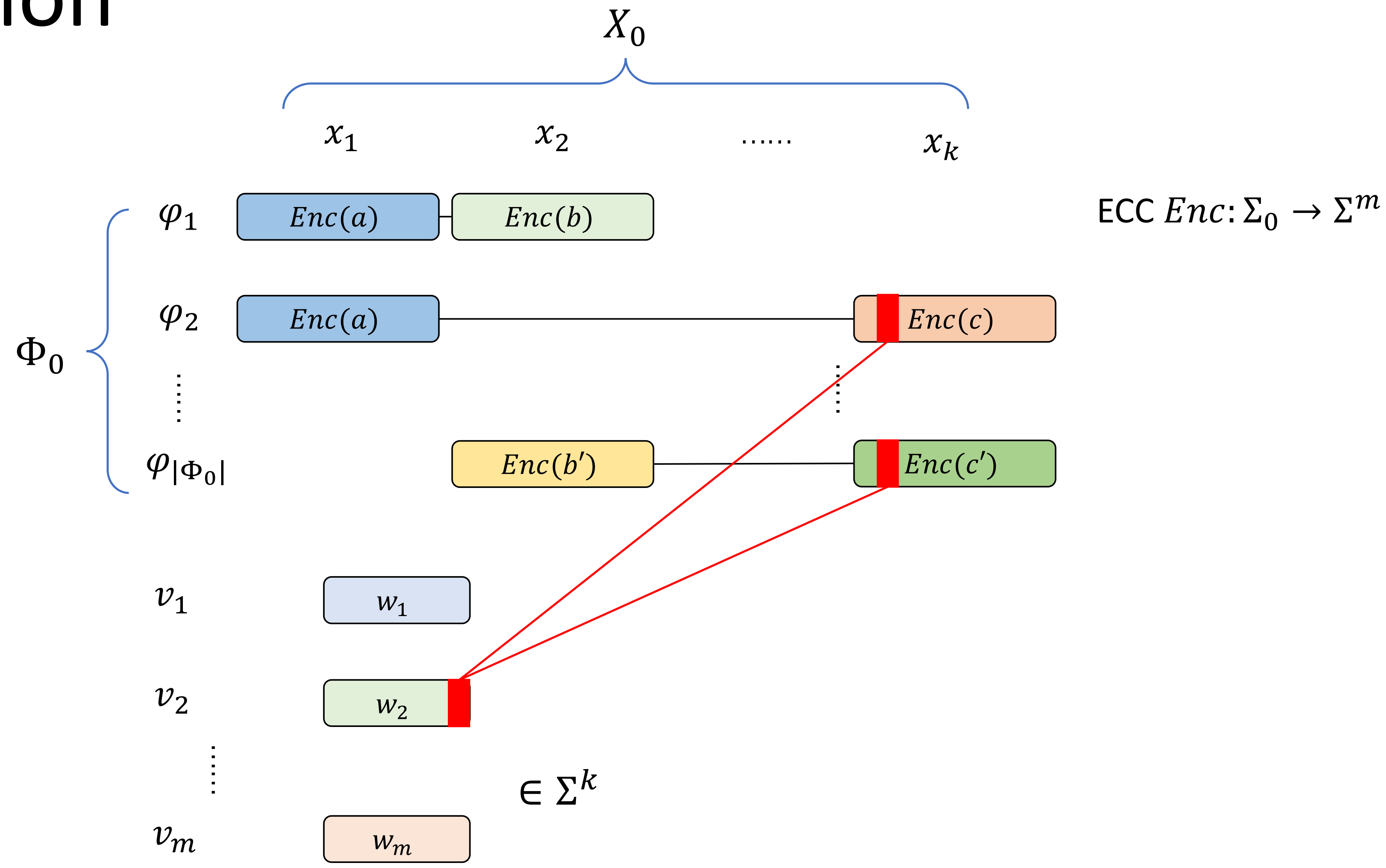

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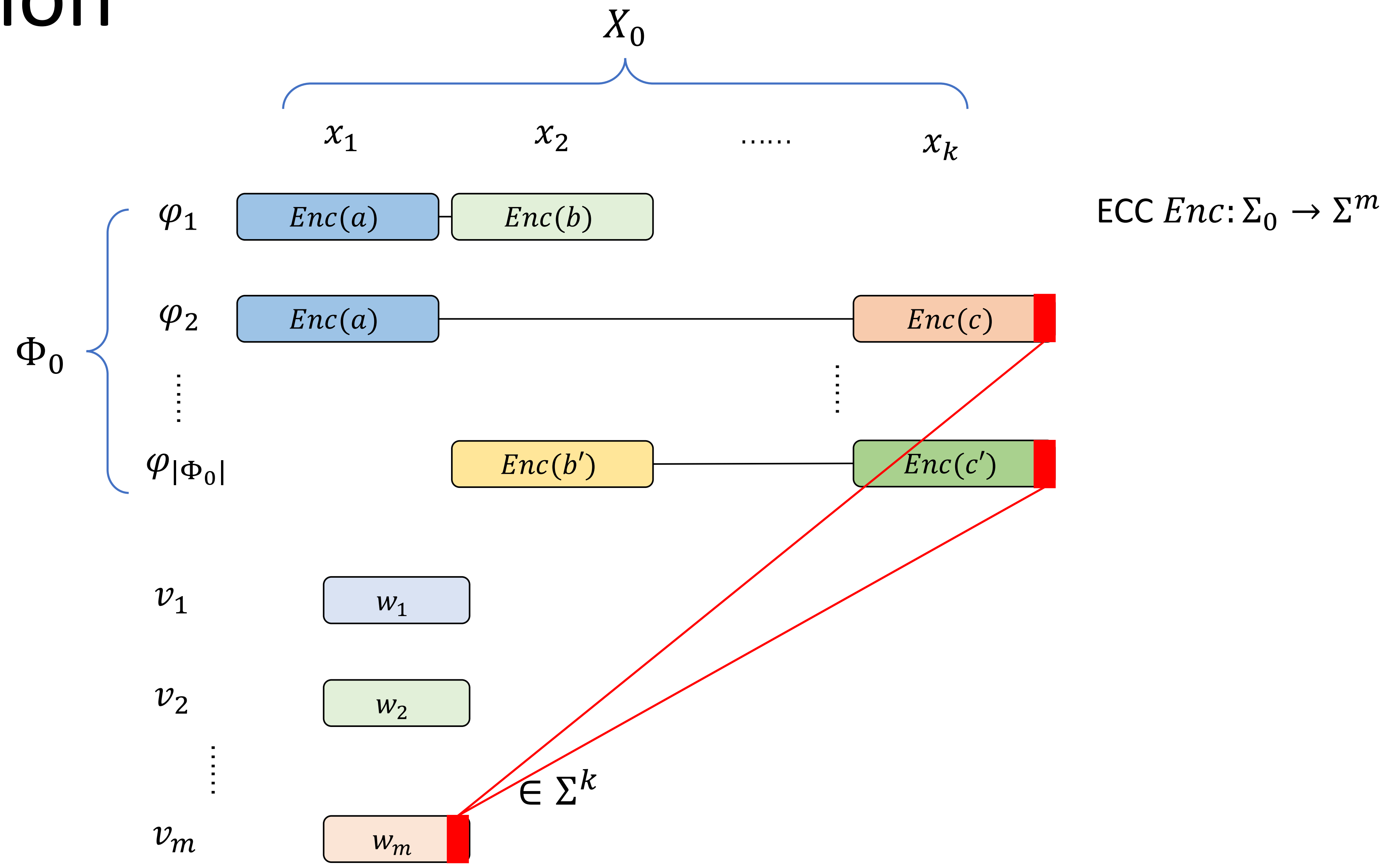
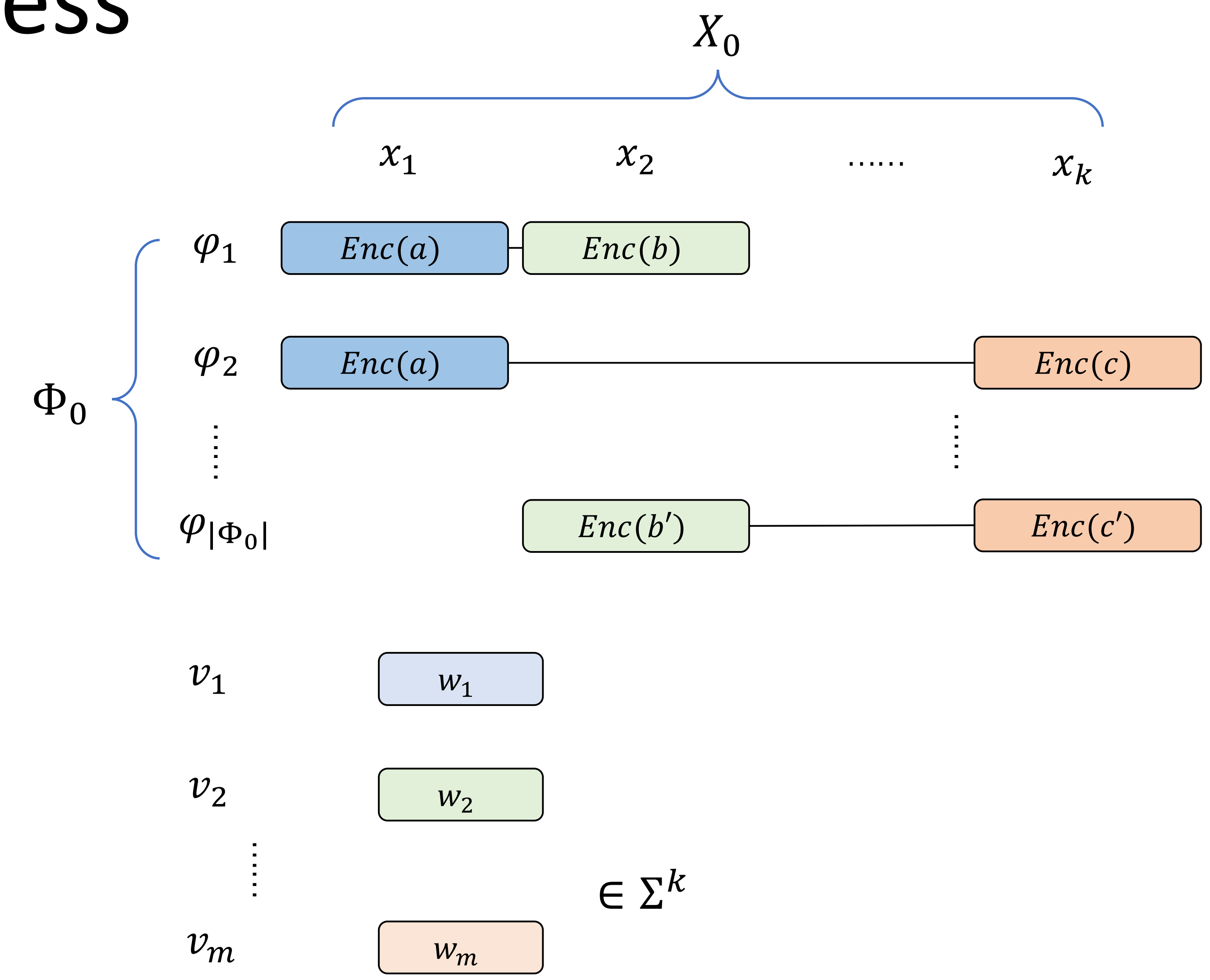


Completeness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Satisfiable

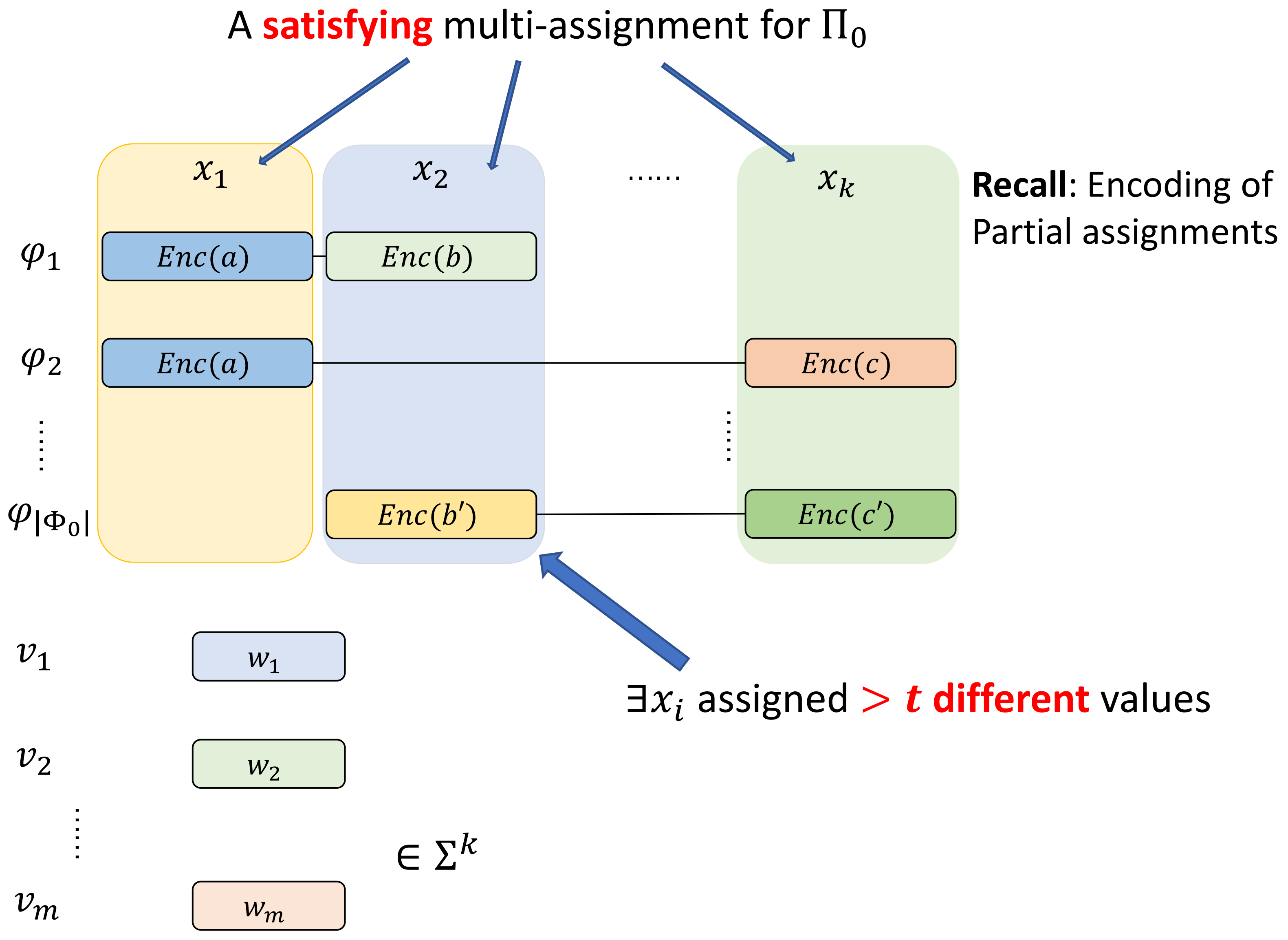
Direct

Π Satisfiable



Soundness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

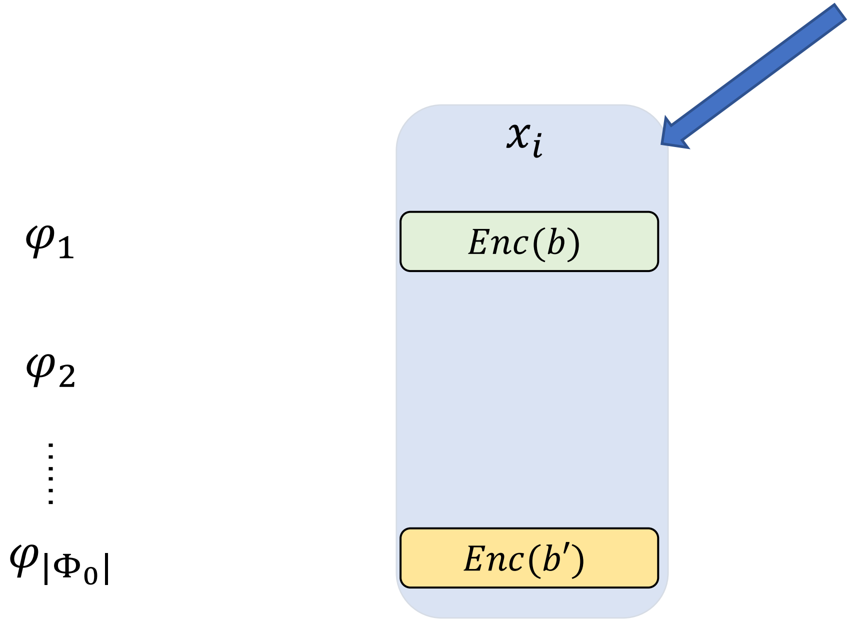


Soundness

$\exists x_i$ assigned $>$ ***t* different** values

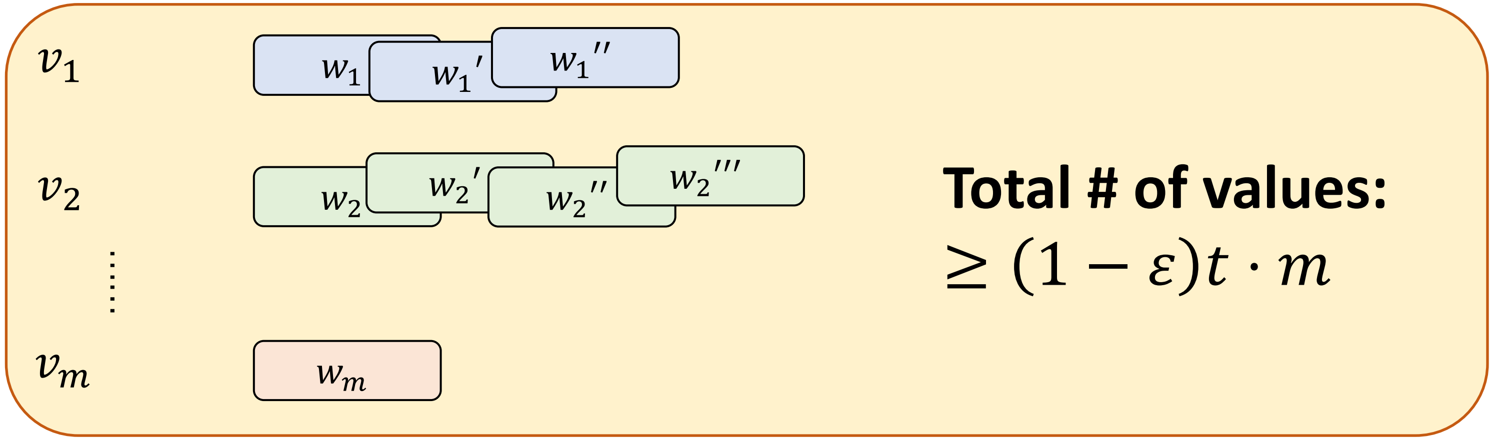
$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$

Can't satisfied when **each** variable assigned $\leq t$ values



Case 1:

More than $(1-\varepsilon)$ fraction of v's, each assigned ***t* + 1 values**



Soundness

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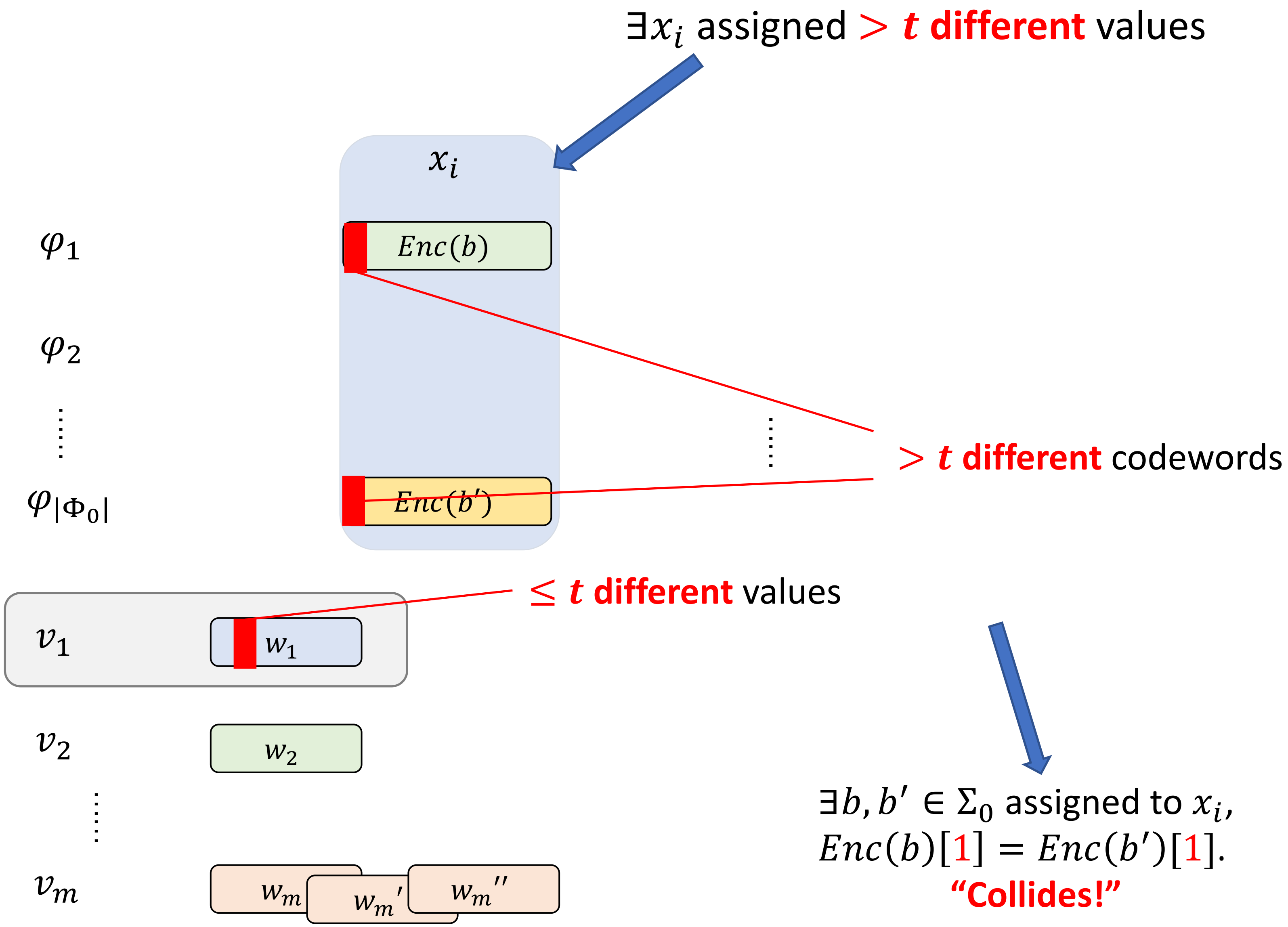
Can't satisfied when **each** variable assigned $\leq t$ values

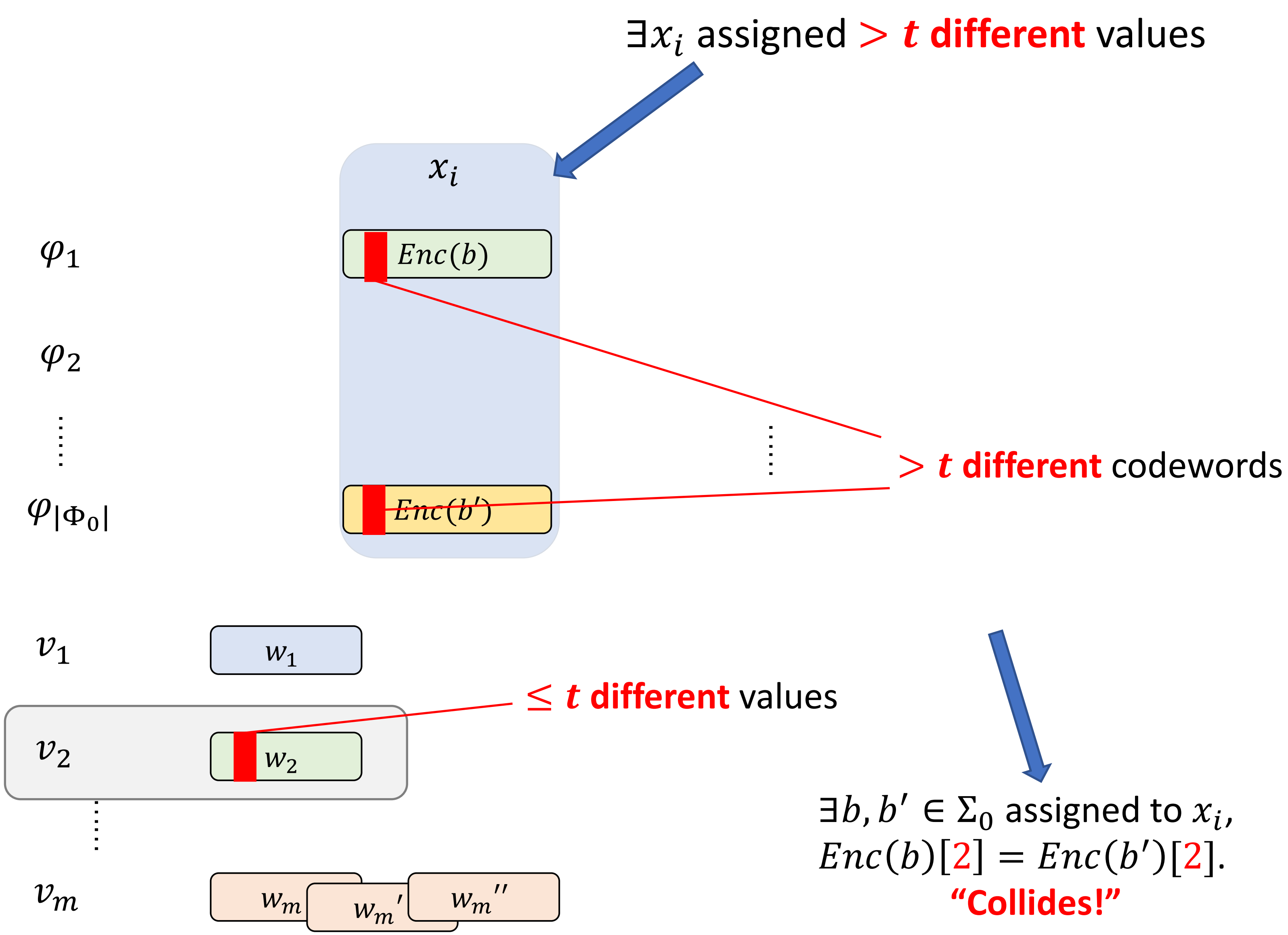


Case 2:
More than ε fraction of v's, assigned **$\leq t$ values**

Soundness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

Case 2:
More than ε fraction of v's, assigned **$\leq t$ values**

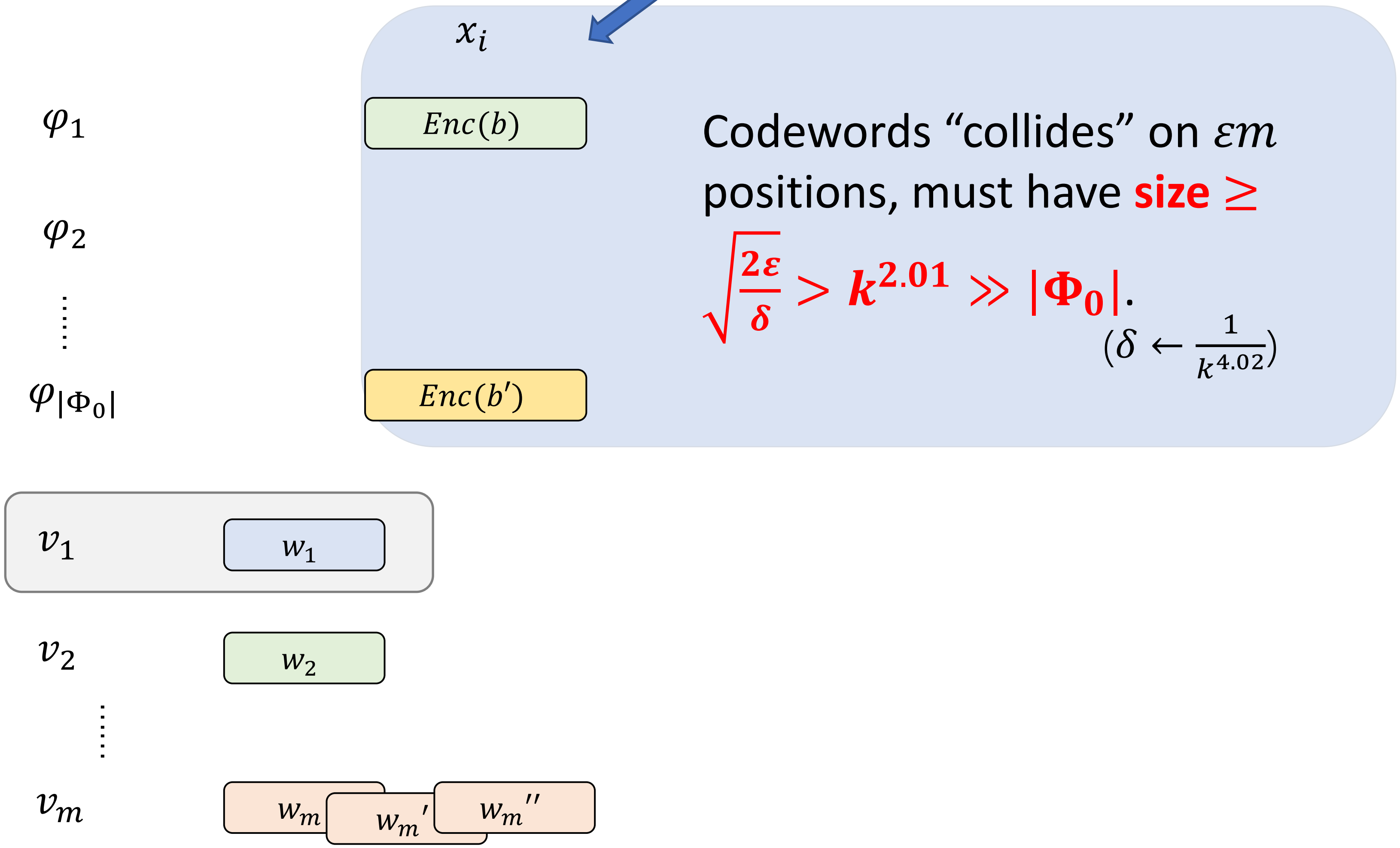


$\exists b, b' \in \Sigma_0$ assigned to x_i,
$Enc(b)[2] = Enc(b')[2]$.
"Collides!"

Soundness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

$\exists x_i$ assigned $>$ ***t* different** values



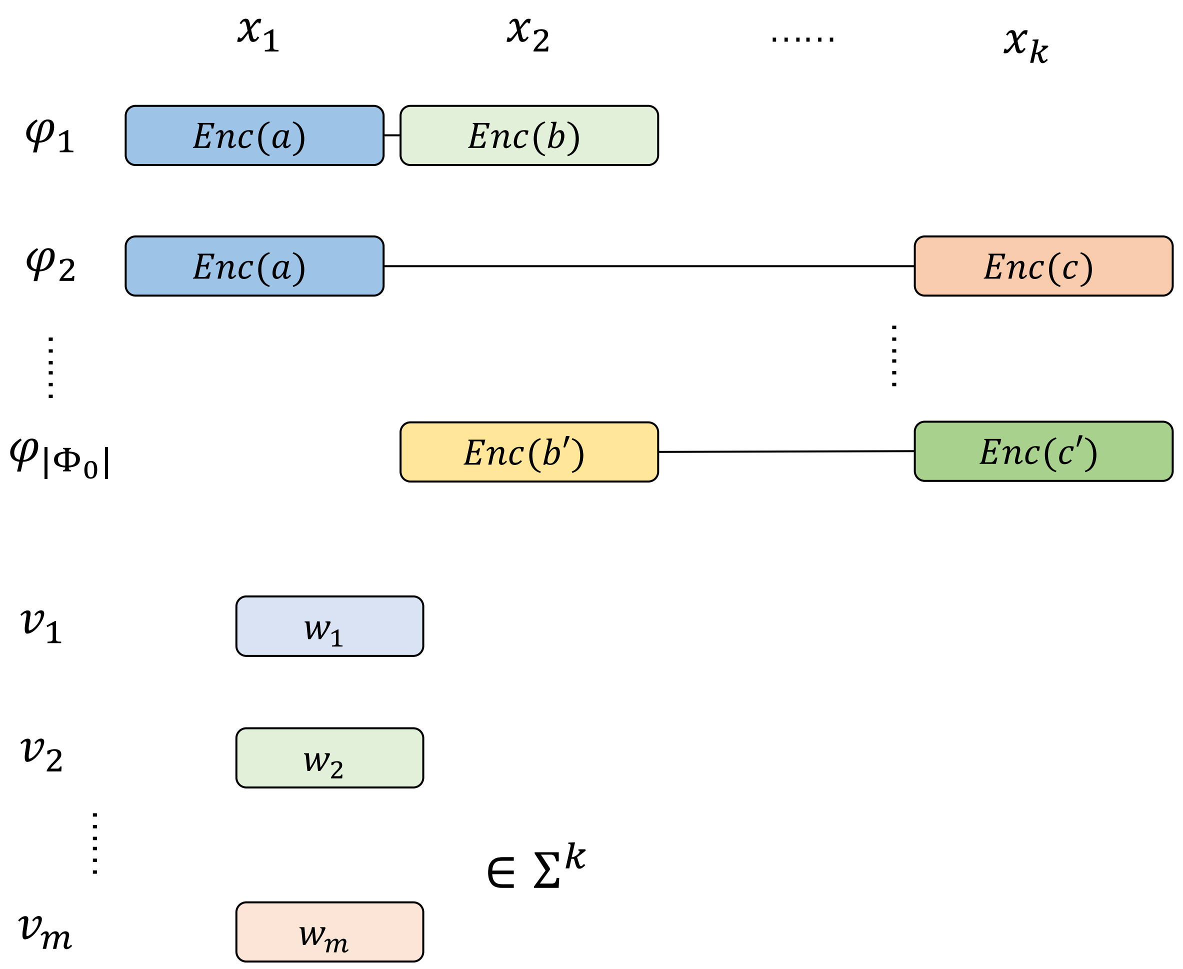
Codewords "collides" on εm positions, must have **size** $\geq \sqrt{\frac{2\varepsilon}{\delta}} > k^{2.01} \gg |\Phi_0|$.
$(\delta \leftarrow \frac{1}{k^{4.02}})$

Case 2:
More than ε fraction of v's, assigned $\leq$ ***t* values**

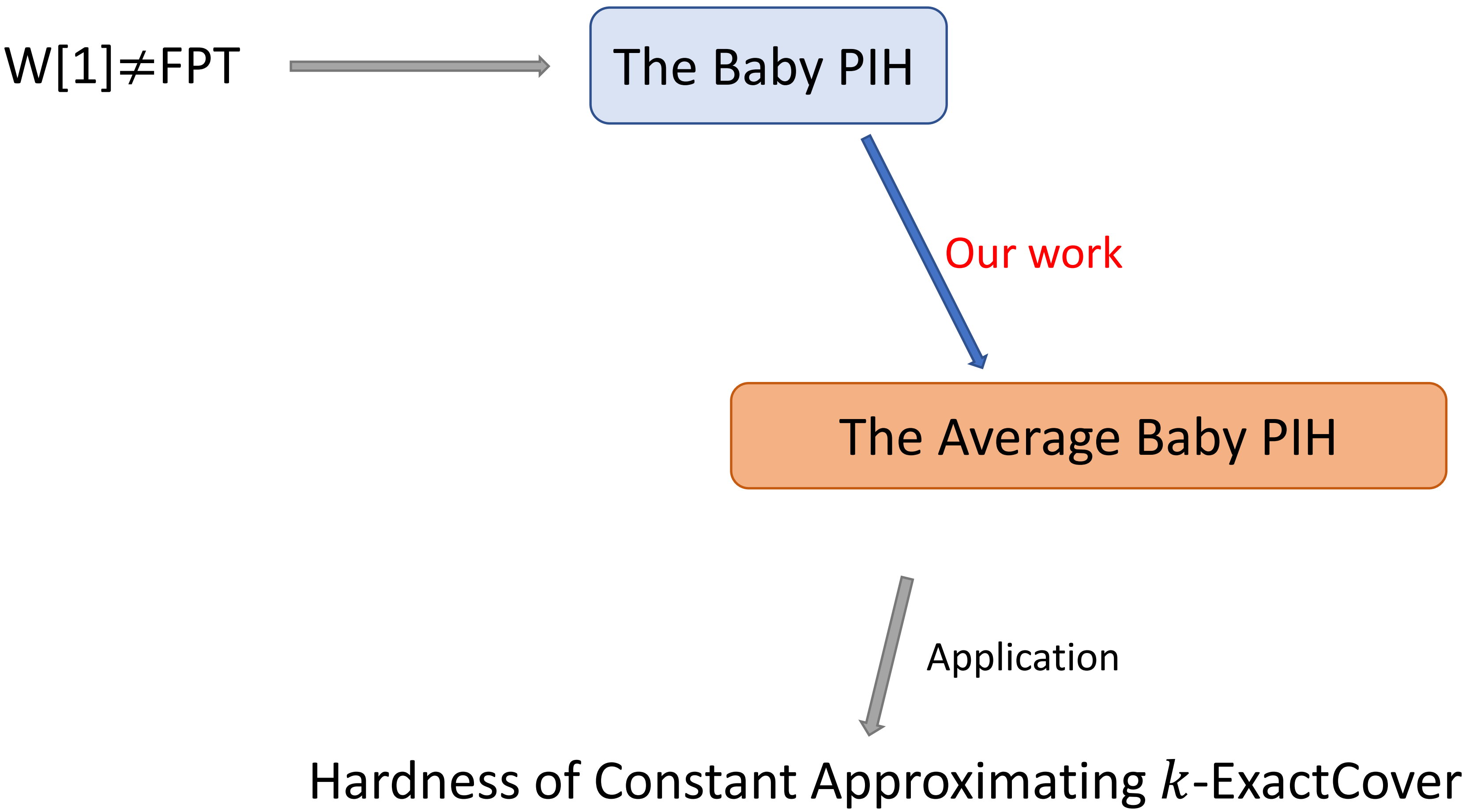
Soundness

$\Pi_0 = (X_0, \Sigma_0, \Phi_0)$
Can't satisfied when **each** variable assigned $\leq t$ values

Π Can't satisfied when assigning to X less than $\min(\frac{t}{2}|X|, k^2)$ values **in total**.



Conclusion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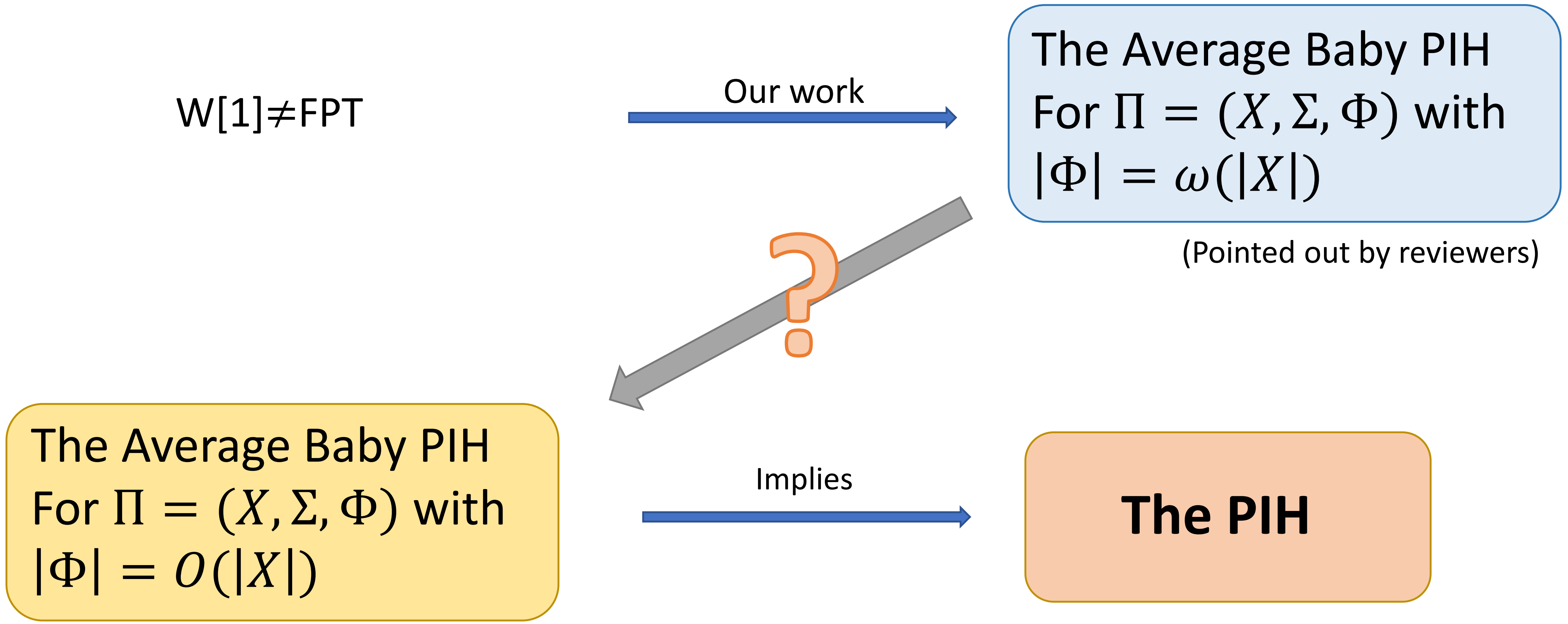
The Baby PIH
Our work
The Average Baby PIH
Application
Hardness of Constant Approximating k-ExactCover

Open Question



Thank You!